# Odd-frequency pairing in a binary mixture of bosonic and fermionic cold atoms

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We study fermionic superfluidity in a boson-single-species-fermion cold-atom mixture. We argue that apart from the standard *p*-wave fermion pairing mediated by the phonon field of the boson gas, the system also exhibits *s*-wave pairing with the anomalous correlator being an odd function of time or frequency. We show that such a superfluid phase can have a much higher transition temperature than the *p*-wave and may exist for sufficiently strong couplings between fermions and bosons. These conditions for odd-frequency pairing are favorable close to the value of the coupling at which the mixture phase separates. We evaluate the critical temperatures for this system and discuss the experimental realization of this superfluid in ultracold atomic gases.

## I. INTRODUCTION

Symmetry of superfluid phases in correlated fermion liquids has been a subject of extensive research for many decades. Two of the most famous examples of systems exhibiting nontrivial symmetries of the order parameter are superfluid $^3$He and high-$T_c$ cuprates, where pairing is believed to occur in *p*-wave triplet and *d*-wave singlet channels, respectively. In 1974, in an attempt to explain the existence of several superfluid phases in $^3$He, Berezinskii[1] had suggested that there is yet another possibility for the triplet pairing: while the *s*-wave component of the anomalous correlator $\langle \psi_\alpha(\mathbf{r},t)\psi_\beta(\mathbf{r}',t') \rangle$ should identically vanish at equal times due to the Pauli principle,[2] it may still be nonzero for $t \neq t'$ (i.e., the anomalous correlator is an odd function of time or frequency) thus giving rise to the superfluid $^3$He *A* phase. While experiments have shown that Berezinskii's conjecture is not realized in the actual $^3$He liquid, the idea of such nonlocal–in-time pairing has attracted considerable attention later with the discovery of superconductivity in cuprates[3,4] and heavy fermion compounds.[5] In particular it has been argued that the nonlocal character of such an order parameter provides a natural resolution to the "paradox" of the coexistence between the strong but instantaneous short-range Coulomb repulsion and Cooper pairing in these materials. More recently the odd pairing mechanism was proposed to explain the anomalous proximity effect in superconductor-ferromagnet junctions,[6] where superconducting penetration length is believed to be significantly enhanced due to the formation of the triplet odd-parity component from the standard *s*-wave singlet condensate.

Despite considerable interest in the subject, the question of whether an odd-frequency phase exists in equilibrium physical systems with no external source of pairs, i.e., as a consequence of spontaneous symmetry breaking, remains unresolved. Previous work has suggested that such phases may be thermodynamically unstable,[7] but the situation remains unclear. Here we discuss the possibility that a boson-fermion mixture, presently realizable in atomic traps, provides an example of a system where an odd-frequency fermionic superfluid phase may exist under the appropriate conditions.

We show that due to the interaction with the phonons in the bosonic subsystem, the fermions at sufficiently low temperatures exhibit pairing either in the *p*-wave channel or in the *s*-wave odd-frequency channel. A key result is that the *s*-wave odd-frequency pairing exists only when the coupling $\gamma$ (to be defined below) between the phonons and the fermions exceeds a certain threshold value $\gamma_c$. Moreover the value of $\gamma_c$ is close to the coupling strength at which the mixture phase separates. That is, upon an increase in the boson-fermion coupling the phonon mode softens, thus leading to stronger attractive interaction between the fermions, as a result of which the odd-frequency fermionic condensate can form in the vicinity of the phase-separation transition. We estimate the transition temperatures for the system. We also point out certain cross correlations between the boson and fermion densities that could possibly be detected in cold-atom mixtures as a signature of the odd-frequency superfluid phase.

## II. MODEL

We describe the dilute mixture of fermions and bosons with the following Hamiltonian density:

$$H = H_B^0 + \frac{\lambda}{2}|\psi_B^\dagger \psi_B|^2 + H_F^0 + \lambda' \psi_B^\dagger \psi_B \psi_F^\dagger \psi_F, \qquad (1)$$

where $H_{B,F}^0$ denote the Hamiltonians for noninteracting bosons and fermions, and $\lambda$ and $\lambda'$ are the boson-boson and boson-fermion coupling constants. We assume that the trap confining the particles is magnetic and therefore the fermions are fully spin polarized. As a result, due to the Pauli principle, direct interaction between fermion atoms is negligible. For the purposes of the present calculation we also neglect the spatial variation of the trapping potential and assume that the local fermion and boson densities are controlled by the chemical potentials so that $H_{B,F}^0 = \psi_{B,F}^\dagger(-\nabla^2/2m_{B,F} - \mu_{B,F})\psi_{B,F}$. Also, here and in the following we set $\hbar = k_B = 1$.

In order to derive an effective coupling between fermions it is convenient to rewrite the bosonic fields in terms of the density-phase variables as $\psi_B = \sqrt{\rho}\exp(i\phi)$. Furthermore,

writing $\rho=\rho_0+\delta\rho$, where $\rho_0$ is a constant and $\delta\rho$ contains only nonzero frequency $\omega_n$ and wave vector $\mathbf{q}$ components of $\rho(\omega_n,\mathbf{q})$, expanding Eq. (1) up to $O(\delta\rho^2,\phi^2)$ around $\rho_0$, and integrating out the phase variable, we obtain the effective "electron"-phonon model with the Matsubara action

$$S_{\text{eff}} = \int_{-\beta/2}^{\beta/2} d\tau \int d^3\mathbf{r}(\mathcal{L}_F^0 + \lambda' \delta\rho\psi_F^\dagger\psi_F)$$
$$+ \sum_{\omega_n,\mathbf{q}} \frac{1}{2}\left(\frac{m_B\omega_n^2}{\rho_0 q^2} + \lambda + \frac{q^2}{4m_B\rho_0}\right)|\delta\rho(\omega_n,\mathbf{q})|^2. \quad (2)$$

In Eq. (2) $\mathcal{L}_F^0$ is free-fermion Lagrangian, $\beta=1/T$, where $T$ is the temperature, and the last term describes phonons or Bogolubov quasiparticles with dispersion relation $\omega=c_sq(1+\xi_0^2q^2)^{1/2}$, where the phonon speed of sound $c_s=(\lambda\rho_0/m_B)^{1/2}$ and the boson coherence length $\xi_0=(1/4m_B\lambda\rho_0)^{1/2}$. The phonons give rise to the nonlocal fermion density-density interaction, with vertex $-\lambda'^2 D^0(\omega_n,\mathbf{q})/2$, where $D^{0-1}(\omega_n,\mathbf{q})$ is the expression in the brackets in the last term in Eq. (2).

It should be noted that while the effective description in Eq. (2) is valid for finite values of the coupling constant $\lambda'$, it breaks down for sufficiently large $\lambda'$ or fermion density. Indeed, by evaluating the renormalization of the phonon Green's function $D^0$ within the second-order (in $\lambda'$) perturbation theory in the particle-hole channel (i.e., accounting for a single fermion polarization bubble in phonon self-energy), we see that

$$D^{0-1} \to D^{-1} = \frac{m_B\omega^2}{\rho_0 q^2} + \lambda(1-\gamma) + \lambda\xi^2q^2 + O(q^4), \quad (3)$$

where $\gamma=\lambda'^2q_F^2/(2\pi^2\lambda v_F)$ and $\xi^2=\xi_0^2+\gamma/12q_F^2$; $q_F$ and $v_F=q_F/m_F$ are the Fermi momentum and velocity, respectively.[8,9] Thus for $\gamma\to 1$ the phonon mode softens at small $q$, signaling that the mixture becomes unstable against the phase-separation transition. The value of $\gamma=1$ corresponds to the line of spinodal decomposition in the mean-field analysis where the instability shows up as a saddle point in the free energy.[10] While the second-order perturbation theory will not be quantitatively accurate for $\gamma$ close to 1, the higher order renormalizations of the phonon propagator do not alter this conclusion on the qualitative level; their effect merely leads to a redefinition of $\gamma$ and $\xi$.[11] As a result effective interaction between fermions increases as one approaches the spinodal point, diverging for $\omega=\mathbf{q}=0$ at $\gamma=1$. It should be pointed out that the phase separation is, presumably, a first-order phase transition,[10,12] and therefore the separated phase becomes thermodynamically more favorable before the spinodal line. However, an estimate of nucleation rates shows that due to extremely low temperatures as well as relatively weak interparticle interactions such nucleation processes are exponentially slow. That is, upon an adiabatic increase of fermion-boson interaction the system will remain in the metastable mixed state up until the absolute instability (spinodal) line, unless the boson gas parameter is comparable to 1 (see Ref. 12).

## III. GAP EQUATION

The onset of the pairing instability corresponds to the appearance of a nonzero anomalous correlator $F(\mathbf{r}-\mathbf{r}',\tau-\tau')=i\langle\mathcal{T}\psi_F(\mathbf{r},\tau)\psi_F(\mathbf{r}',\tau')\rangle$. The self-consistency equation can be readily obtained within the Eliashberg formalism.[8] Following the standard procedure we derive the linearized eigenvalue-type equation for $F(\omega,\mathbf{q})$:

$$G^{-1}(\omega_n,\mathbf{q})G^{-1}(-\omega_n,\mathbf{q})F(\omega_n,\mathbf{q}) = T\sum_{\omega_n',\mathbf{q}'} F(\omega_n',\mathbf{q}')$$
$$\times \frac{\lambda'^2}{2}[D(\omega_n-\omega_n',\mathbf{q}-\mathbf{q}') - D(\omega_n+\omega_n',\mathbf{q}+\mathbf{q}')],$$
$$(4)$$

where $G$ is the fermion Green's function and $\omega_n=\pi T(2n+1)$ is the Matsubara frequency. Note that the appearance of the difference in the brackets on the right-hand side (rhs) of Eq. (4) is due to the presence of fermions with the same spin only; had we considered the usual singlet pairing, the difference would have been removed by the spin part of the Cooper pair wave function. As a result solutions to Eq. (4) should satisfy the antisymmetry property, $F(\omega_n,\mathbf{q})=-F(-\omega_n,-\mathbf{q})$, which, as expected, rules out a possibility of the standard $s$-wave even-in-time pairing.

In order to solve Eq. (4) we must specify the Green's functions $G$ and $D$, which are renormalized in the particle-hole channel according to the Dyson equations $G^{-1}=G^{0-1}-\Sigma$ and $D^{-1}=D^{0-1}-\Pi$, where $\Sigma$ and $\Pi$ are the fermion and phonon self-energies, e.g., Eq. (3). It is well known from the Fermi-liquid theory[8] that near the Fermi surface the renormalized fermion Green's function $G$ can be written as $Z[i\omega_n-v_F^*(q-q_F)]^{-1}$, where $Z$ is the quasiparticle "weight" coefficient ($Z\leq 1$) and $v_F^*=Zv_F$. Then $G^{-1}(\omega_n,\mathbf{q})G^{-1}(-\omega_n,\mathbf{q})=[\omega_n^2+v_F^{*2}(q-q_F)^2]/Z^2$. Next let us expand $F(\omega,\mathbf{q})$ and $D(\omega,\mathbf{q})$ in orbital harmonics using Legendre polynomials $P_l$. For reasons to be specified below we assume that $c_s/v_F$ is small. Then we notice that $D$ is a relatively slowly varying function of $q$ compared to $F$ (which is strongly peaked at $q=q_F$) and therefore we can set both $|\mathbf{q}|$ and $|\mathbf{q}'|$ in the $D$'s on the rhs of Eq. (4) equal to $q_F$. As a result the $D$'s are functions of the angle between $\mathbf{q}$ and $\mathbf{q}'$ only, and with the use of the addition theorem we obtain

$$[\omega_n^2+v_F^{*2}\delta q^2]\mathcal{F}_l(\omega_n,\delta q) = \frac{(\lambda'Z)^2}{2}T\sum_{\omega_n'}\int \frac{q_F^2 d\delta q'}{(2\pi)^2}$$
$$\times \mathcal{F}_l(\omega_n',\delta q')[\mathcal{D}_l(\omega_n-\omega_n') - (-1)^l\mathcal{D}_l(\omega_n+\omega_n')], \quad (5)$$

where $\mathcal{F}_l$ is the partial component of the anomalous correlator $F$ (i.e., with angular momentum $l$), $\delta q\equiv q-q_F$, and $\mathcal{D}_l=\int_{-1}^1 d\cos\theta D(\omega,q_F\sqrt{2-2\cos\theta})P_l(\cos\theta)$.

It is obvious that there are two types of solutions to Eq. (5): $\mathcal{F}_l(\omega)$ with odd $l$ and even in $\omega$ and vice versa, with even $l$ and odd in $\omega$. For both of these solutions the bracket in the rhs of Eq. (5) can be replaced by $2\mathcal{D}_l(\omega-\omega')$ and then we note that Eq. (5) can be cast in the form $\hat{H}_l\mathcal{F}_l=0$, where $\hat{H}_l$ is a "Hamiltonian" of a particle moving in a two-dimensional

external potential $V_l(\tau, x) \sim -2\mathcal{D}_l(\tau)\delta(x)$ (here $\tau$ is Matsubara time and $x$ is Fourier conjugate to $\delta q$) and $\mathcal{F}_l(\beta/2, x)$ $= \mathcal{F}_l(-\beta/2, x) = 0$.

For $\beta \to \infty$, $\hat{H}_l$ has at least one negative eigenvalue $\epsilon_{\infty, l}$ corresponding to the bound state of the particle. Upon an increase of $T$ (i.e., for finite $\beta$) the "energy" $\epsilon_{\infty, l}$ increases, reaching 0 for some particular value of $\beta \equiv \beta_c$. In order to estimate the value of $\beta_c$ we note that for compact $\mathcal{D}_l(\tau)$ the bound-state "wave function" behaves asymptotically as $\sim \exp(-\sqrt{-\epsilon_{\infty, l}}\tau)$, and therefore the boundary conditions, i.e., the finiteness of $\beta$, start to intervene when $\beta \sim 1/\sqrt{-\epsilon_{\infty, l}}$. Moreover, since $(-\epsilon_{\infty, l})^{-1/2}$ is the only relevant "time" scale (in the limit when $\beta^{-1}$ is much smaller then the cutoff frequency $\omega_c$ of $\mathcal{D}_l$; $\omega_c \sim c_s q_F$), the critical temperature $T_{c,l} = 1/\beta_{c,l}$ for channel $l$ is of the order of $\sqrt{-\epsilon_{\infty, l}}$.

Evidently the ground states of $\hat{H}_l$'s describe only the channels with odd $l$ and even $\omega$ dependence of $\mathcal{F}_l$'s. The second type of solutions, even in $l$ and odd in $\tau$, however, corresponds to the first-excited states of $\hat{H}_l$, which, obviously, are odd functions of $\omega$ or $\tau$. While the same considerations hold for the critical temperatures for these solutions as well, there is an important distinction: in the $\beta \to \infty$ limit the former solutions (e.g., bound ground states) always exist for the attractive $V_l$'s and the latter may only exist when the potential $V_l$ is strong enough. Therefore even at zero temperature the odd-frequency (Berezinskii) states exist only if the coupling constant $\lambda'$ or $\gamma$ exceeds a certain threshold $\gamma_c$. Moreover, since at the threshold point the state's "wave function" is delocalized and it localizes for stronger $V_l$, the odd-frequency phase emerges at zero temperature when $\gamma = \gamma_c$ and extends to nonzero temperatures at stronger couplings.

It is easy to estimate this threshold coupling $\gamma_c$ and the critical temperatures from Eq. (5) (see the Appendix). For the $l = 0$ odd-frequency case, the zero-temperature critical coupling $\gamma_c$ can be estimated as the solution to $\gamma_c = (\alpha^2/Z^2)/\ln[1 + \alpha^2/(1 - \gamma_c)]$, where $\alpha = 2q_F\xi$ (see the Appendix). The solutions to this equation are lying in the interval $1/(1 + Z^2) \leq \gamma_c \leq 1$, depending on the value of $\alpha$. In particular, for $\alpha \gg 1$ one finds $\gamma_c \approx 1 - \alpha^2 \exp(-\alpha^2/Z^2)$, and so the width of the superfluid phase (in terms of the dimensionless coupling $\gamma$) is exponentially small. For $\alpha \sim 1$, $1 - \gamma_c$ is *finite* and thus the odd-superfluid phase is most favorable when the effective boson coherence length $\xi$ is comparable with the interfermion distance. The transition temperature can be calculated numerically (see the Appendix), and the results are shown in Figs. 1(a)–1(c) for different values of $\alpha$. Note that for $\alpha \sim 1$ the critical temperature reaches values $\sim c_s q_F$.

For the even frequency $p$-wave case, we consider the effective potential in the $l = 1$ channel using $\mathcal{D}_1$. We note that this is the same as the $p$-wave case studied in Ref. 13 except that we use the renormalized $\mathcal{D}_1$ following from Eq. (3)—this places the $p$ wave on equal footing for comparison to the $s$-wave odd-frequency phase. The $p$-wave phase exists as the coupling $\gamma$ goes to zero, although the critical temperature is exponentially suppressed. The $p$-wave critical temperature is plotted in Figs. 1(a)–1(c) for different values of the parameter $\alpha$ as a dashed line. Note that due to the relative smallness of the effective coupling strength $\mathcal{D}_1(0)$ the critical temperature for the $p$-wave pairing is much lower than that for
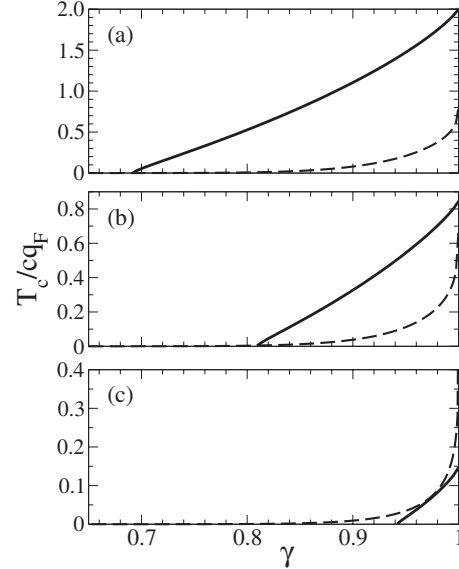


FIG. 1. [(a)–(c)] Critical temperatures vs fermion-phonon coupling $\gamma$ for $\alpha = 2q_F\xi = 1.0$, 1.4, and 2.0. The solid line is the critical temperature for $s$-wave odd-frequency pairing and the dashed line for $p$-wave pairing. In all cases we have set $Z = 1$. The $x$ axis is the same for all three panels, but note the different $y$-axis scales.

the odd-frequency $s$ wave [except for Fig. 1(c), where the odd-frequency region is very small, i.e., for large $\alpha$].

## IV. POSSIBLE EXPERIMENTAL REALIZATION AND DISCUSSION

Since most of the standard (in solid state) thermodynamic and transport measurements are presently unavailable in cold atoms, detecting odd-frequency pairing in a cold-atom fermion-boson mixture may be a nontrivial problem. One intriguing possibility would be to take advantage of the cold-atom time-of-flight type experiments to study the unique correlations of the odd-frequency phase. For example, it has been demonstrated in Ref. 14 that momentum correlations of atomic fermions can be observed by the photodissociation of molecules—upon release from a trap, the atoms exhibit density correlations between points $\mathbf{r}$ and $-\mathbf{r}$ (relative to the center of the trap) as a consequence of the initial molecular state. Similar measurements have been proposed to detect other types of many-body correlations;[15] for example, a fermionic gas in the BCS regime has a density-density correlation function $\langle n_F(\mathbf{r}_1, t) n_F(\mathbf{r}_2, t) \rangle$ which at sufficiently large times of flight $t$ is also peaked at $\mathbf{r}_1 = -\mathbf{r}_2$ as a result of the pairing.

The above argument should, however, be modified when applied to the odd-frequency type of pairing. Indeed, since the equal-time anomalous correlator is identically 0 for the odd-frequency pairing, there is no $\mathbf{r}_1 = -\mathbf{r}_2$ correlation in the two-point fermion correlation function. However, the three-point correlation function $\langle n_F(\mathbf{r}_1) n_F(\mathbf{r}_2) n_B(\mathbf{r}_3) \rangle$, where $n_B$ is boson density, *does* contain the signature of the odd-frequency pairing. To see this one should notice that while $\langle \psi_{F\mathbf{q}}(0) \psi_{F-\mathbf{q}}(0) \rangle = 0$ for such pairing, $\langle \psi_{F\mathbf{q}}(0) \dot{\psi}_{F-\mathbf{q}}(0) \rangle \neq 0$.
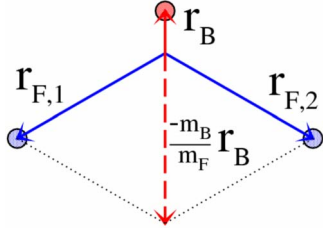
FIG. 2. (Color online) Schematic picture of three-particle correlations under time-of-flight expansion: two fermions, $r_{F,1}$ and $r_{F,2}$, and one boson, $r_B$.

Then, using $i\dot{\psi}_F = [H, \psi_F]$, where $H$ is given by Eq. (1), it is easy to show that $\langle \psi_{F\mathbf{q}_1}(0)\psi_{F\mathbf{q}_2}(0)n_{B\mathbf{q}_3}(0)\rangle \sim \delta(\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3)$, where $n_{B\mathbf{q}}$ is Fourier component of boson density and the width of $\delta$ is primarily controlled by the size of the trap. The boson density can be expressed as $n_{B\mathbf{q}} \approx \sqrt{\rho_0}(\psi_{B\mathbf{q}} + \psi^{\dagger}_{B-\mathbf{q}})$, and therefore the three-point correlation function contains an irreducible contribution peaked at $\mathbf{q}_1 + \mathbf{q}_2 = -\mathbf{q}_3$ (in Fourier space). This three-point correlation can be interpreted as an order parameter proposed in the context of odd-frequency superconductivity in a $t$-$J$ model.[16] As a result the real-space equal-time-correlation function $\langle n_F(\mathbf{r}_1)n_F(\mathbf{r}_2)n_B(\mathbf{r}_3)\rangle$ has a correlation peak at $\mathbf{r}_1 + \mathbf{r}_2 = -(m_B/m_F)\mathbf{r}_3$, where we have used the relationship between the wave vectors of the particles in the initial state with their coordinates in the time-of-flight image, $\mathbf{q}_{F,B} = m_{F,B}\mathbf{r}/t$ (see Fig. 2). Therefore particle density cross correlations which can, in principle, be deduced from instantaneous fermion and boson atom absorption images would provide a direct test for odd-frequency pairing in cold-atom mixtures.

Finally we point out that the validity of Eq. (4) is controlled by the Migdal criterion, i.e., the smallness of the vertex correction part. The latter can be estimated as the three-legged diagram[8] with two fermion and one phonon lines. The standard order of magnitude estimate for the ratio between the bare and the renormalized interaction vertices gives $(\gamma/\sqrt{1-\gamma}) \times (c_s/v_F) \times (q_F\xi)^{-1}$, where we have used the phonon Green's function given by Eq. (3). Thus, Eq. (4) is valid for finite $\gamma$, provided that $\gamma$ is not too close to 1, i.e., the point of the phase separation, and $(c_s/v_F) \times (q_F\xi)^{-1} \ll 1$. Moreover, since we are interested in the regime $q_F\xi \sim 1$, the vertex correction is small as long as $c_s/v_F \ll 1$. Therefore the above results are quantitatively valid when $\xi_0 c_s m_F \ll 1(\hbar)$, that is $m_F/m_B \ll 1$. While this condition is naturally fulfilled in solid-state systems, with $m_F$ and $m_B$ being electron and ion masses, this is not necessarily the case in cold-atom systems. A reasonable choice for testing our theory in trapped cold-atom systems is a $^6$Li-$^{87}$Rb binary mixture so that the mass ratio condition is satisfied. Using the $^{87}$Rb background scattering length of 5.32 nm and a $^6$Li density of $10^{12}$ cm$^{-3}$, the coupling $\gamma = 1$ corresponds to a $^{87}$Rb-$^6$Li scattering length of 22.7 nm, an order of magnitude accessible via an interspecies Feshbach resonance. To optimize the odd-frequency phase, one would want $\alpha = 2q_F\xi \sim 1$ (see Fig. 1). With the same parameters as above, $2q_F\xi_0 = 1$ corresponds to a $^{87}$Rb boson density of $2.3 \times 10^{14}$ cm$^{-3}$ and the temperature scale $T_c \sim c_s q_F \simeq 85$ nK, readily accessible in cold atomic gases.

In summary, we have studied a mixture of bosons and single-species-fermions, showing that fermionic superfluidity of the Berezinskii odd-frequency type is likely to exist under appropriate conditions, i.e., this pairing occurs with a finite critical strength of the boson-fermion coupling. We have estimated the transition temperature and pointed out the unique boson-fermion cross correlations which such a state exhibits.

**APPENDIX**

To evaluate the threshold coupling and the critical temperatures it is convenient to transform the "Schrodinger" equation $\hat{H}_l\mathcal{F}_l = \epsilon_{\infty,l}\mathcal{F}_l$ back to the frequency representation, e.g., Eq. (5), but with the summation over the Matsubara frequencies replaced by integration and with an additional $\epsilon_{\infty,l}\mathcal{F}_l$ term on the left-hand side. Introducing $\Delta_l(\omega, \delta q) = [\omega_n^2 + v_F^{*2}\delta q^2 - \epsilon_{\infty,l}]\mathcal{F}_l(\omega_n, \delta q)$, integrating out the momentum $\delta q'$ on the rhs, and noticing that the solution $\Delta_l$ is independent of $\delta q$ (a result of evaluating the phonon propagator on the Fermi surface), we obtain

$$\Delta_l(\omega) = \frac{(\lambda' Z q_F)^2}{4\pi v_F^*}\int \frac{d\omega'}{2\pi}\frac{\Delta_l(\omega')\mathcal{D}_l(\omega' - \omega)}{\sqrt{\omega'^2 - \epsilon_{\infty,l}}}. \quad (A1)$$

We consider the $l=0$ and $l=1$ solutions to Eq. (A1). Since the effective coupling strength $\mathcal{D}_l(0)$ rapidly decreases for greater $l$, phases with higher orbital momentum of the order parameter have much lower critical temperatures and thus are never realized.

To estimate the critical coupling strength for the $l=0$ odd-frequency solution let us first consider the zero-temperature case [$\epsilon_{\infty,0} = 0$ in Eq. (A1)]. Since $\Delta_0(\omega)$ must be odd, we use the ansatz $\Delta_0(\omega) \propto \omega$ (with cutoff $\omega_c \sim c_s q_F$); we have verified that this linear $\omega$ dependence as $\omega \to 0$ is correct by means of explicit numerical solutions of the gap equation. Expanding the rhs of Eq. (A1) linearly in $\omega$, canceling $\omega$, and integrating the resulting rhs of Eq. (A1) by parts we obtain the condition for the threshold (critical) coupling $4\pi^2 v_F^* \simeq (\lambda_c' Z q_F)^2 \mathcal{D}_0(0)$ [here we have neglected the cutoff dependence assuming that $\mathcal{D}_0(\omega_c) \ll \mathcal{D}_0(0)$]. It is instructive to evaluate the critical coupling strength for $D(0) = D^0(0)$, that is, without accounting for the phonon-mode softening. A straightforward calculation yields $\gamma_c = \lambda_c'^2 q_F^2/(2\pi^2 \lambda v_F) = \alpha_0^2/[Z^2 \ln(1 + \alpha_0^2)]$, where $\alpha_0 = 2q_F\xi_0$. Since this $\gamma_c$ is always greater than 1 (note that $Z \leq 1$), one would conclude that the coupling needed for the formation of the Berezinskii phase is stronger than that of the phase separation ($\gamma = 1$) and, therefore, that the phase does not exist. This conclusion, however, is erroneous because at finite fermion-phonon coupling the renormalization of the phonon propagator, e.g., Eq. (3), is crucial: as the coupling strength approaches that of the phase separation, the effective interaction between fermions increases due to phonon softening. Thus, with $D(0)$ given by Eq. (3) we find that $\gamma_c$ satisfies the equation $\alpha^2/(Z^2\gamma_c) = \ln[1 + \alpha^2/(1 - \gamma_c)]$, where $\alpha = 2q_F\xi$, which has solutions $\gamma_c$

$\leq 1$. In particular, for $\alpha \gg 1$, one finds $\gamma_c \approx 1 - \alpha^2 \exp(-\alpha^2/Z^2)$ and so the width of the superfluid phase (in terms of the dimensionless coupling $\gamma$) is exponentially small. For $\alpha \sim 1$, $1 - \gamma_c$ is *finite* and thus the superfluid phase exists only when the effective boson coherence length $\xi$ is comparable with the interfermion distance. The transition temperature can be estimated within the same linear ansatz for $\Delta_0(\omega)$ by retaining the $-\epsilon_{\infty,0}$ in the denominator on the rhs and solving the resulting equation numerically for $\sqrt{-\epsilon_{\infty,0}} \sim T_{c,0}$, for which the results are presented in Figs. 1(a)–1(c) for different values of $\alpha$.

For the $l=1$ even-frequency $p$-wave phase, to estimate the critical temperature $T_{c,1}$ or $\epsilon_{\infty,1}$ from Eq. (A1), it is sufficient to set $\Delta_1(\omega) = $const (again, with cutoff $\sim c_s q_F$) and replace

$\mathcal{D}_1(\omega - \omega')$ in the rhs of Eq. (A1) by $\mathcal{D}_1(0)$. After a straightforward calculation using the phonon propagator of Eq. (3) we obtain that $T_{c,1} \sim c_s q_F \exp(-1/g_1)$,

$$g_1 = \frac{\gamma}{4\alpha^2}\left[\left(\frac{1-\gamma}{4\alpha^2} + \frac{1}{2}\right)\ln\left(1 + \frac{4\alpha^2}{1-\gamma}\right) - 1\right], \quad \text{(A2)}$$

where, for simplicity, we have set $Z=1$. We note that the $p$-wave phase has been considered in Ref. 14 for fermion-boson mixtures and that our calculation would have been the same as those if we had not used the renormalized phonon propagator. The $p$-wave transition temperature given by Eq. (A2) is plotted in Figs. 1(a)–1(c) for different values of the parameter $\alpha$ as a dashed line.

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